Electrical and Electronics Engineering
2024-2025
Master Semester 2

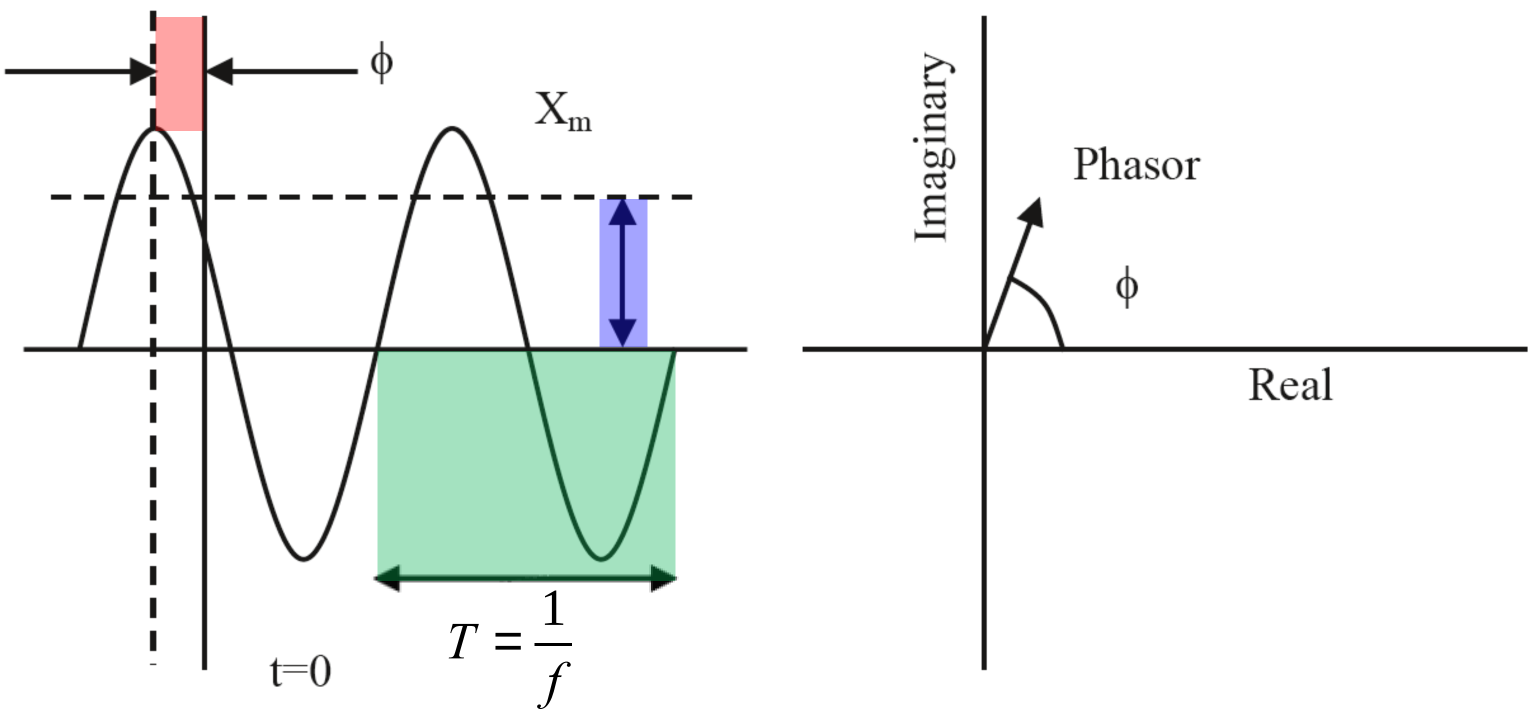
Course
Smart grids technologies

Synchrophasors fundamentals: summary about continuous and discrete Fourier's transformations

Prof. Mario Paolone
Distributed Electrical Systems Laboratory
École Polytechnique Fédérale de Lausanne (Switzerland)

Synchrophasor estimation (basics)

The core component of a Phasor Measurement Unit (PMU) is represented by the **synchrophasor estimation algorithm** whose main task is to **estimate the parameters** of the fundamental tone of a signal (**Amplitude**, **Phase** and **Frequency**) by using a previously acquired set of samples representing a portion of an acquired waveform (i.e., voltages or currents).



What is a phasor?

A phasor X is an element of the complex field $X \in \mathbb{C}$.

$$X = X_r + jX_i$$

Let's use **Eulero's notation**

$$X = |X| \cdot e^{j\psi} = A \cdot e^{j\psi} = A \cdot [\cos(\psi) + j\sin(\psi)]$$

$$|X| = A = \sqrt{{X_r}^2 + {X_i}^2}$$

$$\arg(X) = \psi = tan^{-1}\left(\frac{X_i}{X_r}\right)$$

As well known, phasors are peculiar for representing **time-varying sinusoidal functions**. Let suppose to have a phasor in which its argument varies linearly with time:

$$\arg(X) = \omega t + \varphi = 2\pi f t + \varphi$$

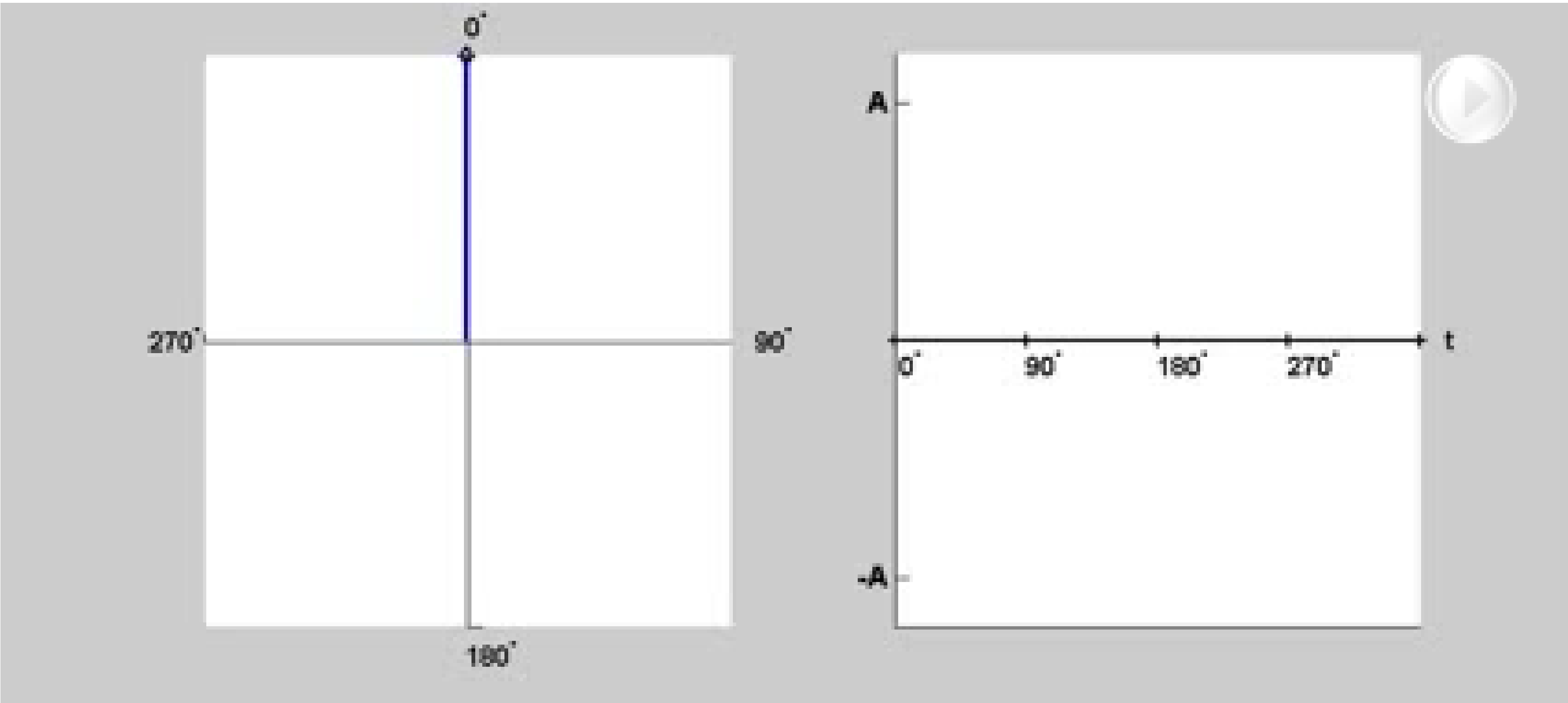
→

What is a phasor?

We get the following **exponential function of time**:

$$X = A \cdot e^{j(\omega t+\varphi)} = A \cdot [\cos(\omega t + \varphi) + j\sin(\omega t + \varphi)]$$

The representation of this function on the complex plane is a **complex number that describes a circular trajectory** with an **angular speed** ω and **radius** A and **initial phase** φ.



What is a phasor?

If we take **discrete photos** of the rotating phasor

$$X = A \cdot e^{j(\omega t + \varphi\ )} = A \cdot [\cos(\omega t + \varphi) + j\sin(\omega t + \varphi)]$$

separated by a time interval Δt such that

$$\Delta t = k\frac{2\pi}{\omega}, k \in \mathbb{N}$$

we will see **the phasor always in the same position on the complex plane**.

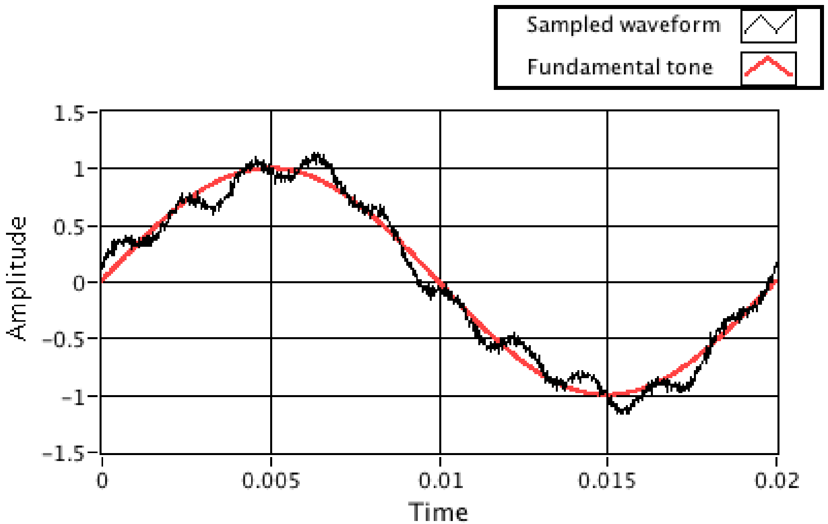
What is a synchrophasor?

Synchrophasor: measurement of the **phasor** of the **fundamental tone** composing a **periodic waveform** of an electric quantity (i.e., voltage, current) using a **common time reference** (usually, the UTC – Universal Time Coordinate).

Fundamental concept:

Extraction of the fundamental tone within a distorted waveform of finite length ➔

Identification of its Amplitude, Phase and Frequency



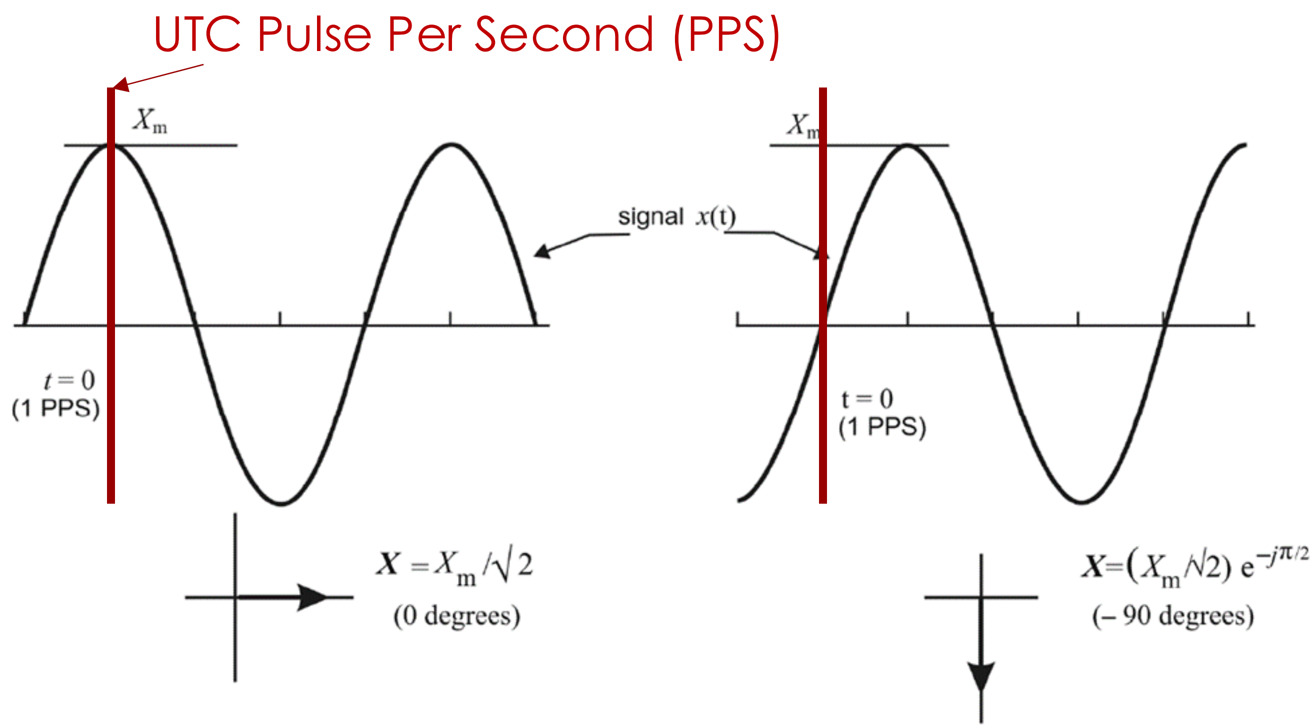
Observation: in the real world the sampled waveforms are always distorted and with time-varying frequency.

What is a synchrophasor?

Synchrophasor: measurement of the **phasor** of the **fundamental tone** composing a **periodic waveform** of an electric quantity (i.e., voltage, current) using a **common time reference** (usually, the UTC – Universal Time Coordinate).

Fundamental concept:

Synchronization with a common time reference



IMPORTANT: the common time reference is fixing the reference phase.

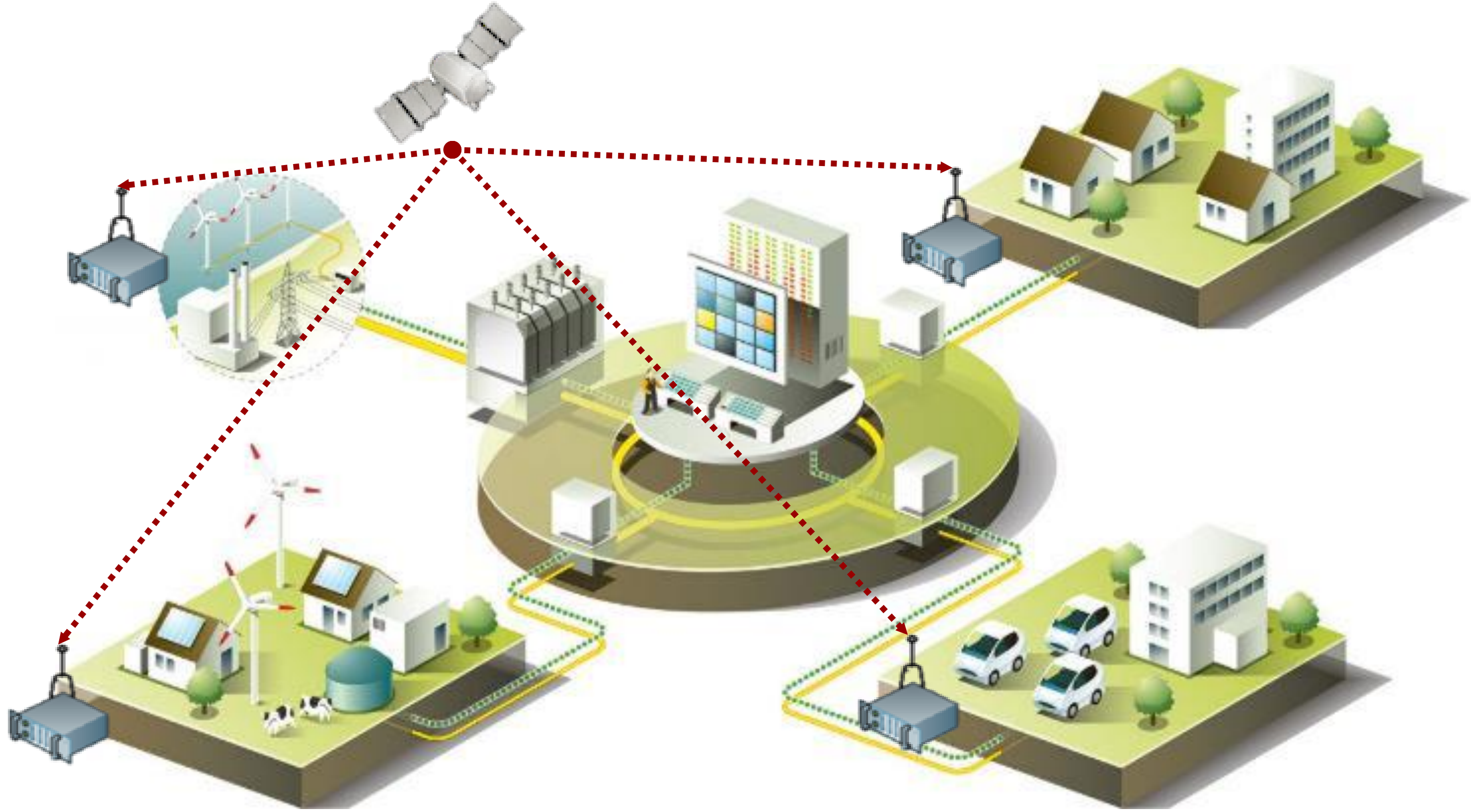
What is a Phasor Measurement Unit (PMU) ?

PMU definition (as stated in IEEE Std.C37.118-2011):

"A device that produces synchronized measurements of ***phasor*** *(i.e. its amplitude and phase),* ***frequency****,* ***ROCOF*** *(Rate of Change Of Frequency) from voltage* ***and/or current signals based on a*** *common time source that typically is the one provided by the Global Positioning System UTC-GPS."*

Why synchronized measurements are useful ?

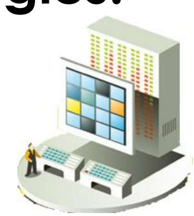
PMU-based power-system monitoring



Involved components/technologies:

Synchronization source

Phasor Measurement Unit (PMU)



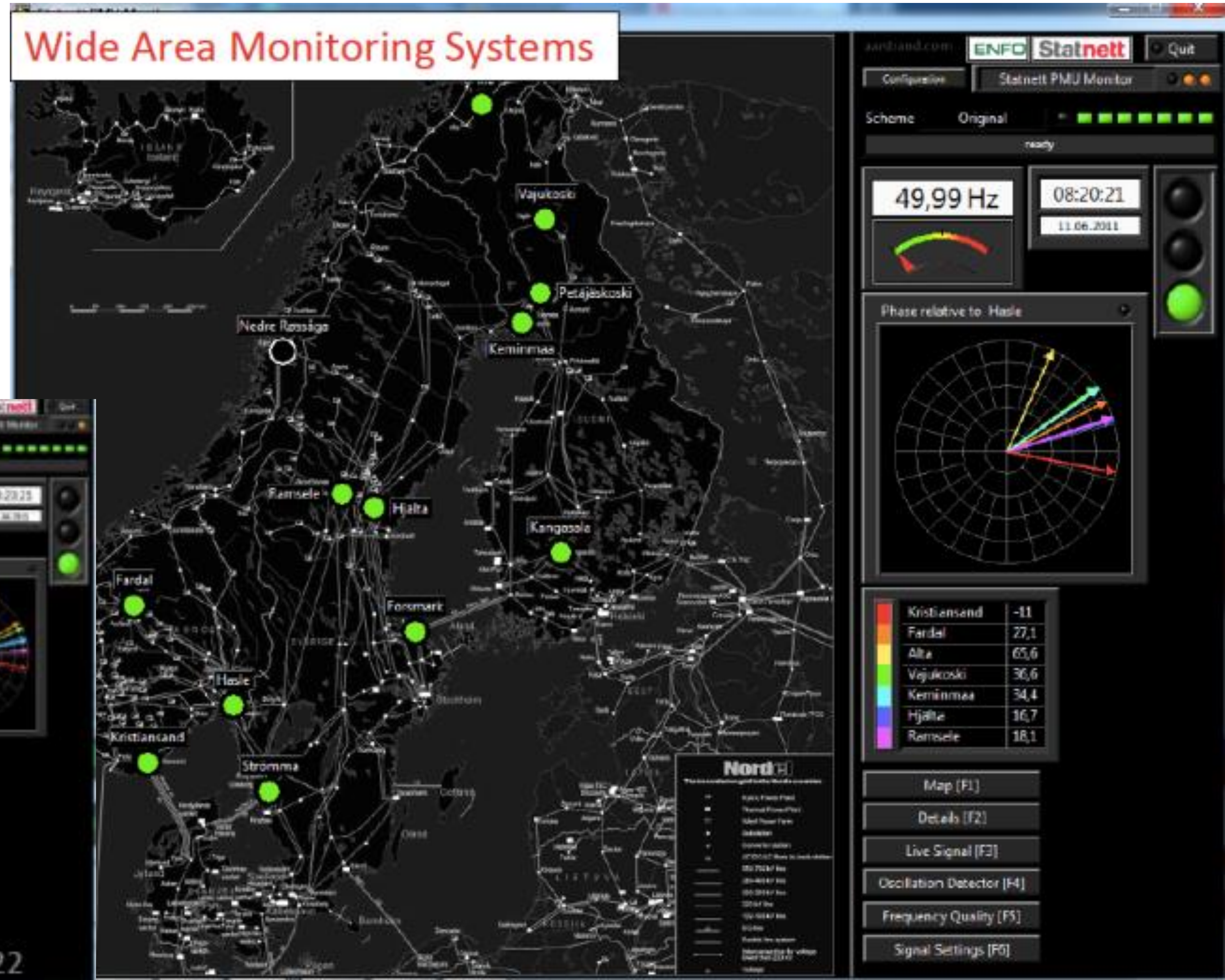
Phasor Data Concentrator (PDC)

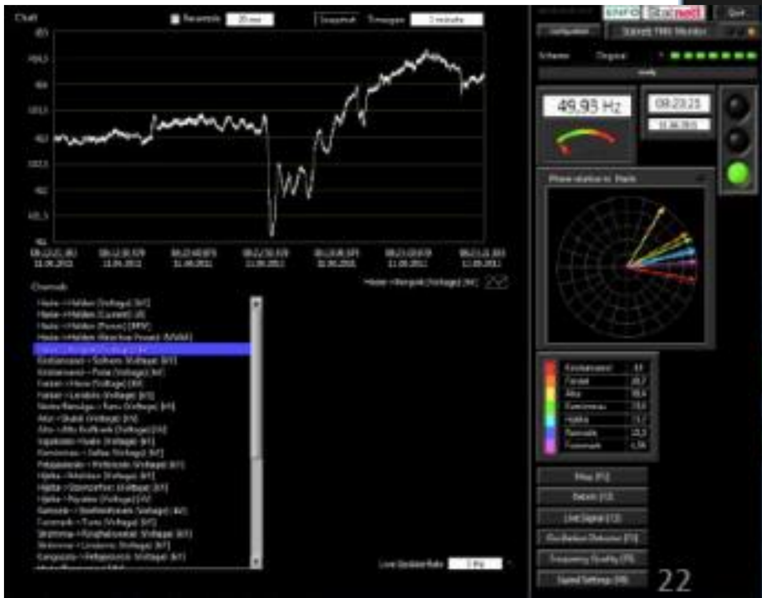
Why synchronized measurements are useful ?

PMU applications

- WAMS – Wide Area Monitoring Systems
- Power systems stability
- Inter-area oscillations
- Voltage stability
- Relaying
- State estimation

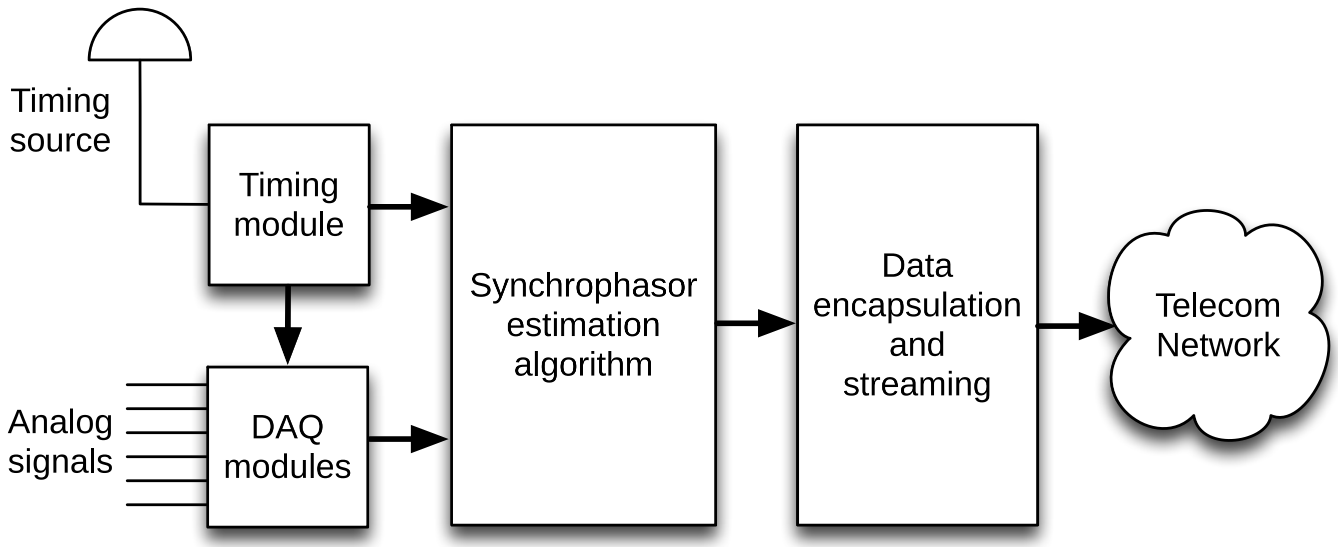
Courtesy of Stattnett



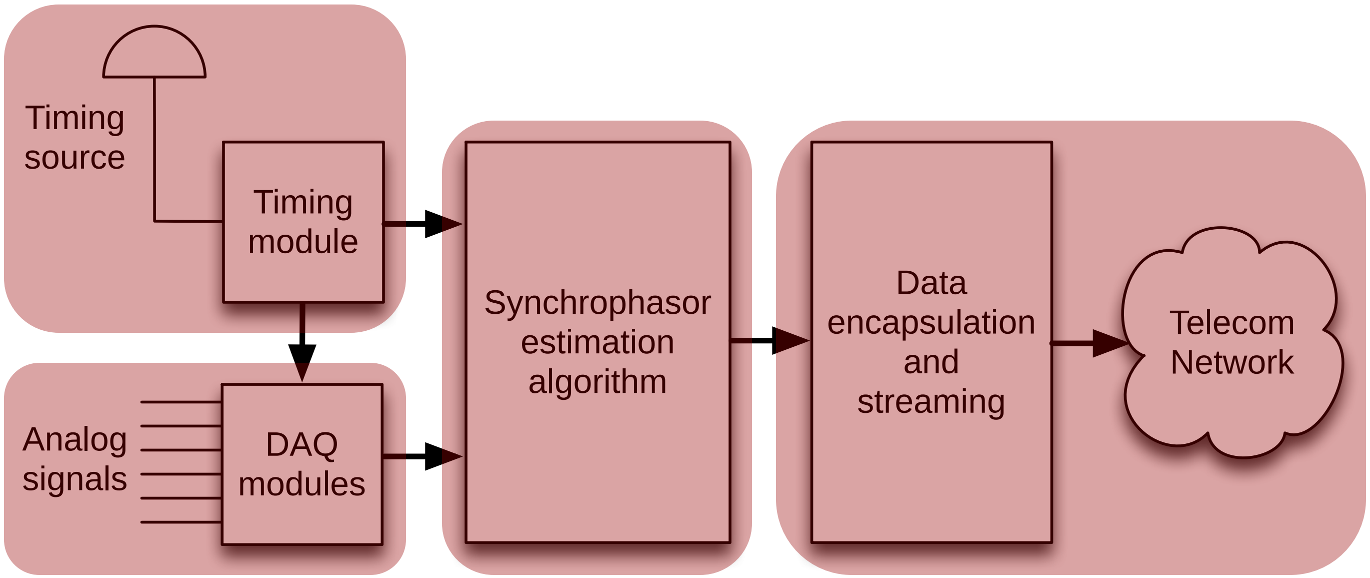


How does a PMU look like?

How does a PMU look like?



PMU components



Timing module:

- Synchronization of different PMUs with a common UTC-time reference
- Several available technologies (GPS, IRIG-B, IEEE 1588, etc.)

DAQ modules:

- A/D conversion of the analog waveforms (voltages and/or currents)
- Sampling process can be free running or synchronized with the UTC-time reference

Synchrophasor estimation algorithm:

- Real-time estimation of frequency, ROCOF and synchrophasor for every input channel
- Several approaches are available in literature

Data encapsulation and streaming:

- According to the data formats specified in the IEEE Std. C37.118-2011 or IEC 61850-90-5

Approaches for the identification of the fundamental frequency tone

Class	Typical algorithms	Advantages	Drawbacks
DFT based	Fourier analysis (e.g., [1])	Low computational complexity, harmonic rejection	Spectral leakage, Harmonic interference
	Interpolated DFT (e.g., [2])		
Wavelet based	Recursive wavelet (e.g., [3])	Harmonic rejection	Computational complexity
Optimization based	WLS (e.g., [4])	They usually provide accurate estimates in combination with other methods	Non deterministic: driven by optimality criteria
	Kalman Filter (e.g., [5])		
Taylor series based	Dynamic Phasor (e.g., [6])	It intrinsically reflects the dynamic behaviors of power systems	Computational complexity

Problem statement

Compute the fundamental frequency component (or main tone) of a time-discrete signal $x(t)$ of unknown characteristics sampled each T_S over a finite number of samples, say N.

Problems to be solved:

1. **How to compute the frequency components (spectrum) of $x(t)$ assuming it is a continuous function.**
2. **How to compute the main tone of $x(t)$ assuming it is a discrete infinite time series.**
3. **How to compute main tone of $x(t)$ assuming it is a discrete and finite time series.**

Basic signal processing concepts

Within the above-listed categories of synchrophasor estimation processes, **we will focus on the DFT-based ones.**

For this reason, we will **recall some basic signal processing concepts.**

Here we briefly illustrate only those topics which are of **direct interest for phasor estimation in power systems.**

The Fourier transform

The Fourier transform of a **continuous time function** $x(t)$ satisfying the integrability conditions is given by

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt$$

Basic signal processing concepts

The **inverse Fourier transform** recovers the time function from its Fourier transform

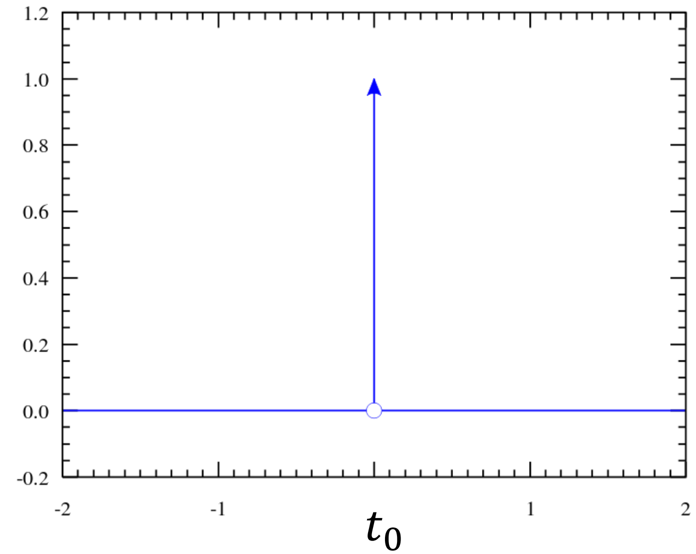
$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi f t} df$$

Sampling and Dirac function

An important function frequently used in calculations using sampled data is the **impulse function** defined as

$$\delta(t - t_0) = \begin{cases} 0 & t \neq t_0 \\ \infty & t = t_0 \end{cases}$$

$$\int_{t_1}^{t_2} \delta(t) dt = 1, t_0 \in [t_1, t_2]$$



Basic signal processing concepts

The impulse function is a **sampling function** in the sense that when the integration of the following equation is performed, the result is the sampled value of the function $x(t)$ at $t = t_0$:

$$x(t_0) = \int_{-\infty}^{+\infty} \delta(t - t_0)x(t)dt$$

The integrals in the form of the above, are known as **convolutions**.

Thus, the **sampling process** at uniform time intervals T_S can be considered to be **a convolution of the input signal and a string of impulse functions** $\delta(t - kT_S)$ where $k \in \mathbb{Z}$ (i.e., integer ranging from $-\infty$ to $+\infty$).

Basic signal processing concepts

Interesting property of the **Fourier transform of the impulse function:**

Forward Fourier transformation of the Dirac — Inverse Fourier transformation of the Dirac

$$\underbrace{\int_{-\infty}^{+\infty} \delta(t-t_0)e^{-j2\pi f t}dt = e^{-j2\pi f t_0}}_{\text{Forward}} \leftrightarrow \underbrace{\int_{-\infty}^{+\infty} e^{-j2\pi f t_0}e^{j2\pi f t}df = \int_{-\infty}^{+\infty} e^{j2\pi f(t-t_0)}df = \delta(t-t_0)}_{\text{Inverse}}$$

Another interesting property involving the impulse function is the **Fourier transform of the complex exponential**

$$\cos(2\pi f_0 t) + j\sin(2\pi f_0 t) = e^{j2\pi f_0 t}$$

$$\int_{-\infty}^{+\infty} e^{j2\pi f_0 t}e^{-j2\pi f t}dt = \int_{-\infty}^{+\infty} e^{-j2\pi(f-f_0)t}dt = \delta(f-f_0)$$

The **convolutions of two time functions** and their Fourier transforms have a **convenient relationship**. Consider the convolution $z(t)$ of two time functions $x(t)$ and $y(t)$

$$z(t) = \int_{-\infty}^{+\infty} x(\tau)y(\tau - t)d\tau \triangleq x(t) * y(t)$$

Property #1

The Fourier transform of a convolution is equal to the product of the Fourier transform of the functions being convolved, or

$$z(t) = x(t) * y(t) \rightarrow Z(f) = X(f) \cdot Y(f)$$

and similarly, the inverse Fourier transform of a convolution of two Fourier transforms is the product of the corresponding inverse Fourier transforms

$$Z(f) = X(f) * Y(f) \rightarrow z(t) = x(t) \cdot y(t)$$

Basic signal processing concepts

Let us consider the functions:

$$x(t) = \cos(\omega t)\,, y(t) = \sin(\omega t)$$

The Fourier transform of $x(t)$ (with $\omega = 2\pi f_0$) is (recall the Fourier transformation of the complex exponential we have seen on slide 23):

$$X(f) = \int_{-\infty}^{+\infty} \cos(2\pi f_0 t) e^{-j2\pi f t} dt = \int_{-\infty}^{+\infty} \left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \right) e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} \frac{e^{-j2\pi (f - f_0) t} + e^{j2\pi (f + f_0) t}}{2} dt = \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

Similarly for $y(t)$ we obtain

$$Y(f) = \frac{1}{2} j[\delta(f + f_0) - \delta(f - f_0)]$$

→

Basic signal processing concepts

Result: the Fourier transforms of a ***pure cosine function*** *of unit amplitude is a* ***pair of real impulse functions in frequency domain located at*** $\pm f_0$ *and that of a* ***pure sine function*** *of unit amplitude is a* ***pair of imaginary impulse functions of opposite signs at*** $\pm f_0$.

Basic signal processing concepts

Property #2

The Fourier transform of a periodic function with period T is a series of complex amplitude impulse functions of frequency $1/T$.

Let $x(t)$ be a periodic function of t, with a period equal to T. As known, it can be expressed as an exponential Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j\frac{2\pi kt}{T}}$$

$$\alpha_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi kt}{T}} dt\, , k = 0, \pm 1, \pm 2, \dots$$

→

Basic signal processing concepts

The Fourier transform of $x(t)$ expressed in the exponential form is given by

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt = \int_{-\infty}^{+\infty} \left[\sum_{k=-\infty}^{\infty} \alpha_k e^{j\frac{2\pi k t}{T}} \right] e^{-j2\pi f t} dt =$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} \alpha_k e^{j\frac{2\pi k t}{T}} e^{-j2\pi f t} dt = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} \alpha_k e^{j2\pi t\left(\frac{k}{T} - f\right)} dt$$

Observation: note that in **the above the order of the summation and integration has been reversed (<u>assuming that this is permissible</u>).**

→

Basic signal processing concepts

By setting $f_0 = 1/T$ (i.e., the **fundamental frequency of the periodic signal**), the integral of the exponential term in the last form is the impulse function $\delta(f - kf_0)$, and thus the Fourier transform of periodic $x(t)$ is:

$$X(f) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} \alpha_k e^{-j2\pi t(f-kf_0)} dt = \sum_{k=-\infty}^{\infty} \alpha_k \delta(f - kf_0)$$

Observation: the above is a **series of impulses located at multiples of the fundamental frequency** f_0 of the periodic signal with **impulse magnitudes being equal to amplitude of each frequency component in the input signal**: we have therefore obtained the **discrete signal spectrum**.

Basic signal processing concepts

Discrete Fourier Transform: it is a method to **calculate the Fourier transform of a discrete (limited) number of samples (say N) sampled each T_S from an unknown input signal** $x(t)$.

The Fourier transform is calculated at **discrete steps in the frequency domain**, just as the input signal is sampled at discrete instants in the time domain. Let's see how.

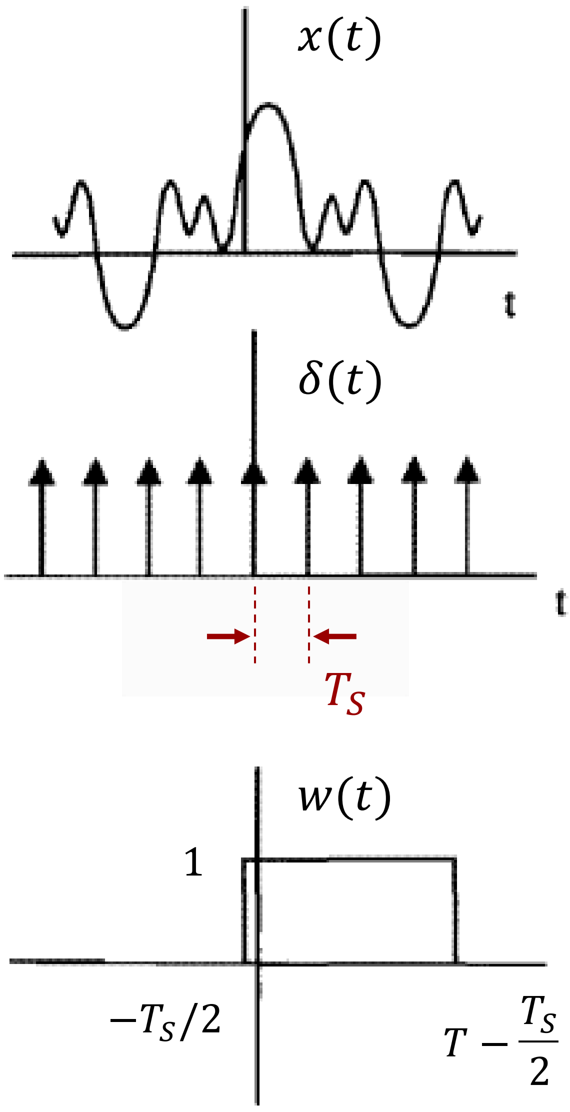
Consider the process of selecting N samples from a time-domain function $x(t)$ separated by a discrete time T_S ➔ $x(kT_S)$ with $k = 0,1, \dots, N-1$ (**note that this time k has a limited range**).

This process is equivalent to multiplying the sampled data train by a "windowing function" $w(t)$ ***that, in this case, is a rectangular function of time with unit magnitude and a span of*** $T = NT_S$

→

Basic signal processing concepts

With the choice of samples ranging from 0 to $N-1$, it means that the windowing function can be viewed as starting at $-T_S/2$ and ending at $(N-1/2)T_S = T - T_S/2$



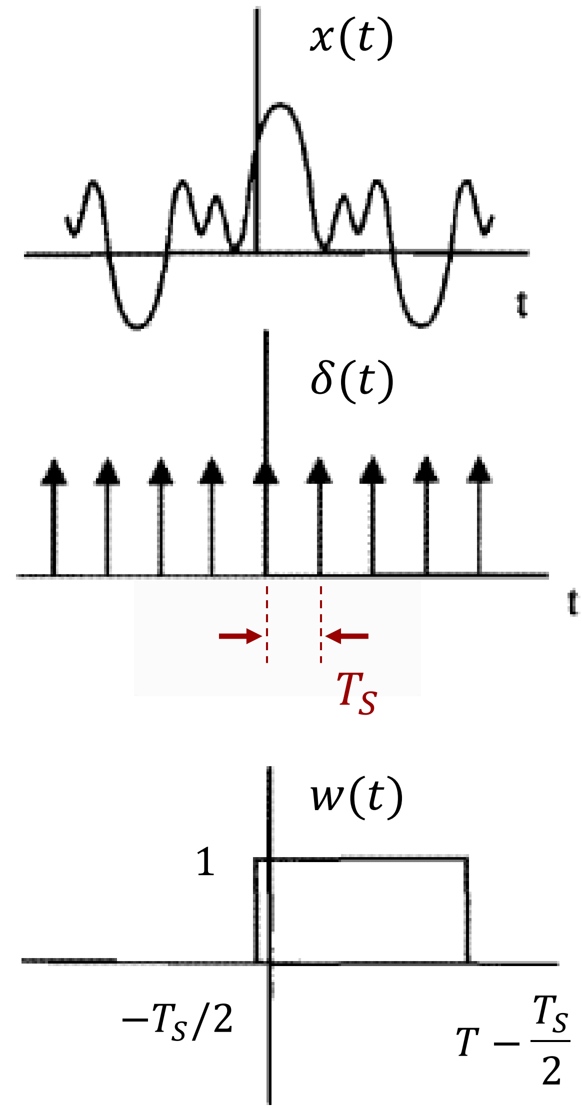
Basic signal processing concepts

These samples can be viewed as being obtained by the multiplication of the signal $x(t)$ by the sampling function $\delta(t)$ and by the windowing function $w(t)$:

$$y(t) = x(t)\delta(t)w(t) = \sum_{k=0}^{N-1} x(kT_S)\delta(t-kT_S)$$

We remind what already introduced regarding the use of the impulse function:

$$x(t_0) = \int_{-\infty}^{+\infty} \delta(t-t_0)x(t)dt$$



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Basic signal processing concepts

As a consequence, **the Fourier transform of the sampled windowed function $y(t)$ is then the convolution of the Fourier transforms of the three functions** $x(t), \delta(t), w(t)$.

Problem: how can we account for $w(t)$?

➔ The Fourier transform of $y(t)$ is to be **sampled in the frequency domain** in order to obtain the DFT of $y(t)$. Let's see what this statement means.

Using property #2 applied to $w(t)$ supposed to be periodic, the **discrete** frequency domain of $w(t)$ contains multiples of $1/T$, **where T is the span of the windowing function**. We can introduce the **frequency sampling function** $\Phi(f)$ and its Fourier inverse as:

$$\Phi(f) = \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \leftrightarrow \phi(t) = T \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

→

Basic signal processing concepts

In order to **obtain the samples in the frequency domain**, we must multiply the Fourier transform $Y(f)$ with $\Phi(f)$.

To obtain the **corresponding time domain function** $x'(t)$ we will require a convolution in the time domain of $y(t)$ and $\phi(t)$:

$$x'(t) = y(t) * \phi(t) =$$

$$= \left[\sum_{k=0}^{N-1} x(kT_S)\delta(t - kT_S)\right] * \left[T \sum_{n=-\infty}^{\infty} \delta(t - nT)\right] =$$

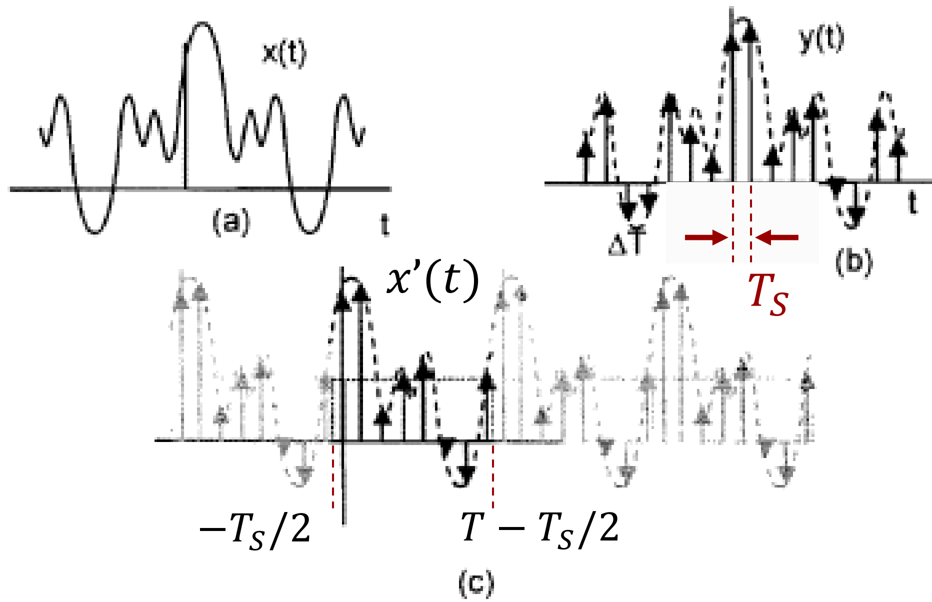
$$= T \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} x(kT_S)\delta(t - kT_S - nT)\right]$$

→

Basic signal processing concepts

The function $x'(t)$ is periodic with a period T.

Observation: the windowing function $w(t)$ limits the data to samples 0 through $N-1$, and the sampling function $\phi(t)$ transforms the original N samples in time domain to an infinite train of N samples with a period T as shown in figure (c).



Note that, **although the original function $x(t)$ was, in general, not periodic, the function $x'(t)$ is periodic and we may consider this function to be an approximation of $x(t)$.**

→

The Fourier transform of the periodic function $x'(t)$ is a **sequence of impulse functions in frequency domain.**

By using the **property #2**, we get:

$$X'(f) = \sum_{n=-\infty}^{\infty} \alpha_n \delta\left(f - \frac{n}{T}\right)$$

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} x'(t)\, e^{-j\frac{2\pi n t}{T}} dt\,, n = 0, \pm 1, \pm 2, \ldots$$

Substituting for $x'(t)$ in the above expression, we get:

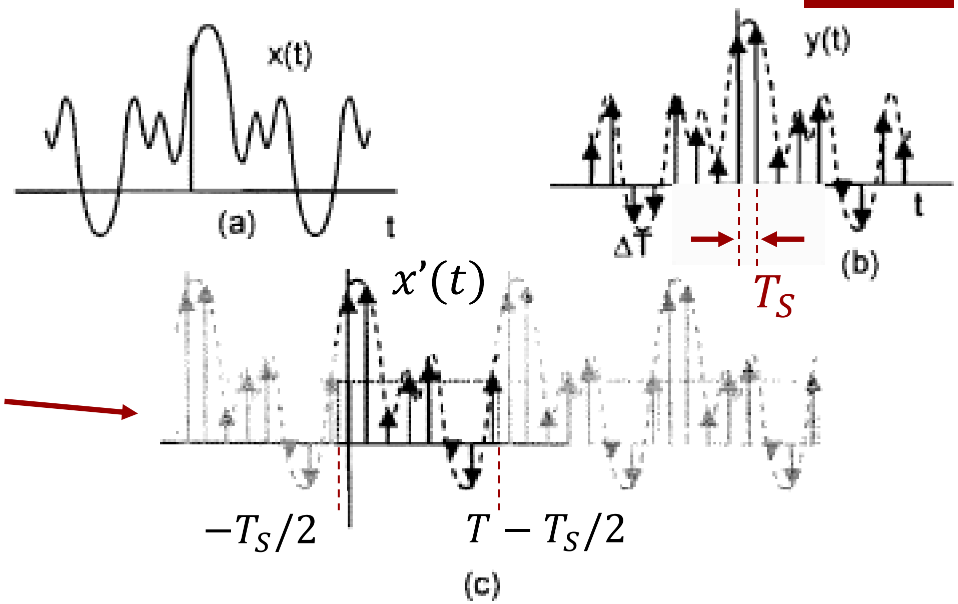
$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} \left\{T \sum_{m=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} x(kT_S)\delta(t - kT_S - mT)\right]\right\} e^{-j\frac{2\pi n t}{T}} dt$$

$$m = 0, \pm 1, \pm 2, \ldots$$

→

Basic signal processing concepts

In the previous equation, the index m designates the train of periods of figure (c)



Since the limits on the integration span is of one period only, we may remove the summation on m, and set $m = 0$, thus using only the samples over the period shown in bold in figure (c). Therefore, the previous equation becomes

$$\alpha_n = \int_{-T/2}^{T/2} \sum_{k=0}^{N-1} x(kT_S)\delta(t - kT_S)\, e^{-j\frac{2\pi nt}{T}} dt = \sum_{k=0}^{N-1} x(kT_S) \int_{-T/2}^{T/2} \delta(t - kT_S) e^{-j\frac{2\pi nt}{T}} dt$$

$$= \sum_{k=0}^{N-1} x(kT_S) e^{-j\frac{2\pi knT_S}{T}}, n = 0, \pm 1, \pm 2, \ldots$$

→

Basic signal processing concepts

Since we remind that there are N samples in the data window T, namely $NT_S = T$, we have:

$$\alpha_n = \sum_{k=0}^{N-1} x(kT_S)e^{-j\frac{2\pi kn}{N}}, n = 0, \pm 1, \pm 2, \dots$$

Although the index n goes over all positive and negative integers, it should be noted that there are only N distinct coefficients α_n. Thus, α_{N+1} is the same as α_1, and the Fourier transform $X'(f)$ has only N distinct values corresponding to frequencies $f = n/T$, with n ranging from 0 through $N-1$:

$$X'\left(\frac{n}{T}\right) = \sum_{k=0}^{N-1} x(kT_S)e^{-j\frac{2\pi kn}{N}}, n = 0,1,2, \dots, N-1$$

→

The previous equation is the **definition of the DFT of N input samples, sampled at equidistant discrete intervals T_S.**

The DFT is symmetric about $N/2$, the components beyond $N/2$ simply belong to negative frequency. Thus the DFT does not calculate frequency components beyond $N/2$, which also happens to be the Nyquist limit to avoid aliasing errors (see the next lecture).

References

1. A. G. Phadke and J. S. Thorp, Synchronized Phasor Measurements and Their Applicationm New York: Springer, 2008